DISCUSSION OF AN UNIQUENESS THEOREM BY L. N. SLOBODETSKY AND M. I. CHRAMOV, FOR SOLUTIONS TO THE CAUCHY PROBLEM FOR A QUASILINEAR SYMMETRICAL SYSTEM OF DIFFERENTIAL EQUATIONS

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DISCUSSION OF AN UNIQUENESS THEOREM BY L. N. SLOBODETSKY AND M. I. CHRAMOV, FOR SOLUTIONS TO THE CAUCHY PROBLEM FOR A QUASI-LINEAR SYMMETRICAL SYSTEM OF DIFFERENTIAL EQUATIONS

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Discussion of an Uniqueness Theorem

by L. N. Slobodetsky and M. I. Chramov,

for Solutions to the Cauchy Problem for a Quasi-Linear

Symmetrical System of Differential Equations.

by

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The generalized Cauchy problem is succinctly explained by I. G. Petrovsky ([1], page 26). Briefly, we are given a system of N equations with N unknown functions $u_1, u_2, \cdots, u_N$

$$\varphi_i\left\{(x_o, x_1, \cdots, x_n, u_1, \cdots, u_N, \cdots) \frac{\partial^k u_j}{\partial x_o^{k_o} \partial x_1^{k_i} \cdots \partial x_n^{k_n}}, \cdots\right\} = 0$$

for $(i, j = 1, 2, \cdots N)$

and are required to find a solution for $u_1, u_2, \cdots u_N$ of this system in some neighborhood of a smooth n-dimensional surface E, this solution to satisfy certain prescribed initial conditions. In their paper, ([2]), Slobodetsky and Chramov consider the question of uniqueness of a solution to this Cauchy problem for particular quasi-linear systems of the 1st and 2nd order. Because their systems contain a certain type of symmetry, the question of uniqueness is answered more easily by a direct examination of the system itself, than by an attempt to

reduce it to the question of uniqueness of a solution to a related linear system, in the manner of Hadamard ([3]).

We will put most of our attention on the symmetric system of the 1st order. We say that the system

$$\frac{\partial k}{\partial t} = \sum_{i=1}^{n} \sum_{j=1}^{N} a_{jk}^{(i)} \frac{\partial u_j}{\partial x_i} + f_k \quad (k = 1,2,\cdots,N) \tag{1}$$

(where $a_{jk}^{(i)} = a_{jk}^{(i)}(t,x,u)$ and $f_k = f_k(t,x,u)$ are complex functions of the real variables t and x, and the unknown vector-function u_j; $x = (x_1,\cdots, x_n)$ and $u = (u_1,\cdots, u_N)$) is symmetric in some domain D of the (n+N+1)-dimensional space (t,x,u), if in D we have

$$a_{jk}^{(i)} = \bar{a}_{kj}^{(i)} \quad (i = 1,2,\cdots,n; \; j,k = 1,2,\cdots,N) \tag{2}$$

The domain D, if it contains points M(t,x,u), must also contain the points $\bar{M}(t,x,\bar{u})$ where $\bar{u} = (\bar{u}_1,\cdots,\bar{u}_N)$. It is further assumed that the domain is bounded by the plane t = 0, and lies in the region where $0 \leqq t$, for physical considerations.

Slobodetsky and Chramov showed that a solution to system (1) is unique in a certain subdomain of D if the following conditions are satisfied:

1. The system has the symmetry described above.

2. The functions $a_{jk}^{(i)}$ are continuous and have continuous derivatives in D, with respect to the variables $x_1,\cdots,x_n$, $u_1,\cdots, u_N$.

3. The vector function f satisfies the inequality

$$|f(t,x,u) - f(t,x,v)| \leqq \Phi(|u-v|) \qquad (3)$$

when (t,x,u) and $(t,x,v) \in D$.

4. $F(Z) \equiv \sqrt{Z}\ \Phi(Z)$ is a positive, increasing, and concave function, such that

$$\int_0^{\delta} \frac{dZ}{F(Z)} \text{ diverges, } 0 < \delta \qquad (4)$$

The quantity $|u-v|$ is defined as follows:

$$|u-v| \equiv (u-v,\ u-v)^{1/2}$$

where the scalar product of two vectors $Z = (Z_1, \cdots, Z_N)$ and $y = (y_1, \cdots, y_N)$ is given by

$$(Z,y) = \sum_{j=1}^{N} Z_j\ \bar{y}_j.$$

Thus

$$|u-v| = \left\{ \sum_{j=1}^{N} (u_j - v_j,\ \overline{u_j - v_j}) \right\}^{1/2}$$

F(Z) is called a concave function in some interval (a,b) if for any $Z_k \in (a,b)$ $(k = 1, 2, \cdots, m)$ and for any non-negative l_k $(k = 1, 2, \cdots, m)$ such that

$$\sum_{k=1}^{m} l_k = 1, \text{ we have the inequality}$$

$$\sum_{k=1}^{m} l_k\ F(Z_k) \leqq F\left(\sum_{k=1}^{m} l_k Z_k \right)$$

It is remarked that these conditions of uniqueness of the solution for such a system appear

similar to the conditions of Osgood ([4], page 344), as discussed by J. Tamarkine ([5]) for the uniqueness of the solution of the Cauchy problem for a system of ordinary differential equations. The condition of Lipschitz for the uniqueness of a solution to the equation $\frac{dy}{dx} = f(x,y)$ is the well known inequality

$$|f(x,y_2) - f(x,y_1)| \leqq K\ |y_2-y_1|.$$

In his paper ([4]), Osgood extended the Lipschitz condition in the following manner. Given a continuous function $\omega(u)$ such that

$$\omega(0) = 0; \quad 0 < \omega(u),\ 0 < u; \ \omega(-u) = \omega(u)$$

$$\text{and} \quad \lim_{\epsilon \to 0} \int_{\epsilon}^{u_o} \frac{du}{\omega(u)} = \infty$$

then if

$$|f(x,y_2) - f(x,y_1)| \leqq \omega(y_2-y_1) \text{ for } |a-x| \leqq \rho,$$

$|b-y_1| \leqq S$, $|b-y_2| \leqq S$, then at the point $x = a$, the solution will take the value b and this solution will be unique. Tamarkine reformulated this condition by noting that if

$$|f(x,y_2) - f(x,y_1)| \leqq \omega(|y_2-y_1|)$$

where ω is a continuous, positive, increasing function, and $\omega(0) = 0$, then the solution is unique if,

$$\text{for } 0 < u < u_o, \quad \lim_{u \to 0} \int_{u}^{u_o} \frac{du}{\omega(u)} = \infty$$

We shall now attempt to demonstrate the proof that these conditions stated by Slobodetzky and Chramov

do indeed insure uniqueness in the very particular case when $n = 1$ and $N = 2$. The system (1) can then be written out explicitly as

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= a_{11} \frac{\partial u_1}{\partial x} + a_{21} \frac{\partial u_2}{\partial x} + f_1(x,t,u) \\ \frac{\partial u_2}{\partial t} &= a_{12} \frac{\partial u_1}{\partial x} + a_{22} \frac{\partial u_2}{\partial x} + f_2(x,t,u) \end{aligned} \qquad (5)$$

which of course may be also written in matrix form

$$\left(\frac{\partial u}{\partial t}\right) = (A) \left(\frac{\partial u}{\partial x}\right) + (f) \qquad (6)$$

where $A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$

This symmetrical system (5) is of the hyperbolic type described by Petrovsky ([1] page 53) because its characteristic equation

$$|(a_{jk}\alpha) - \lambda \delta_{jk}| = 0 \qquad (7)$$

has only real roots for any real α. This is readily seen by expanding the determinant to get

$$\lambda^2 - (a_{11} + a_{22})\alpha\lambda - (a_{11}a_{22} + a_{12}a_{21})\alpha^2 = 0$$

By the symmetry property assigned to this system we note that a_{11} and a_{22} are real, and that $a_{12}\, a_{21} = |a_{12}|^2$. Thus the discriminant is $\alpha^2[(a_{11}-a_{22})^2 + 4|a_{12}|^2]$ which, for a real α, is always greater than zero. Since no restrictions have been placed on the multiplicity of roots of the characteristic equation (7), this system does not, generally speaking, belong to the class

"hyperbolic in the narrow sense" discussed by Petrovsky ([1] page 59).

Because of the symmetry property assigned to (6), the matrix A is Hermitian, or self-adjoint, and there is associated with it the Hermitian form

$$H(y:\alpha) \equiv (A\alpha y, y) \qquad (8)$$

where $(A\alpha y,y) = \sum_{j=1}^{N} (A\alpha y)_j \, \bar{y}_j$ is again the scalar product of the vectors $A\alpha y$, and y. Since the a_{jk} are to be continuous on a closed domain to be described below, we can say that $|a_{jk}| < q$ and $|\lambda_N| \leqq L_A$ for $= 1$, where λ_N are the roots of equation (7). It is easily shown that $L < Mq$, where M is an absolute constant independent of the coefficients of system (6). We may further point out a property of the Hermitian form,

$$|H(y:\alpha) \leqq L\, |y|^2 \alpha \qquad (9)$$

where $|y| = \sqrt{(y_1 y)}$ is the length of the vector y.

The domain $\Omega(t)$ in the 2-dimensional space (t,x) is called "fundamental" for equation (6) if it is restricted to be between $t = 0$ and $0 < t = T$ and is bounded by a smooth curve E, the exterior normal to which forms an acute angle $(\bar{n},t)$ with the t-axis, such that $\tan(\bar{n},t) \leqq \frac{1}{L}$. We further denote by $S(t)$ any section of $\Omega(T)$ by a line t = constant. See figure 1. We will show that for any solutions $u(t,x)$ and $v(t,x)$ which satisfy the prescribed initial conditions, and such that $(t,x) \in \Omega(T) \Rightarrow (t,x,u)$ and $(t,x,v) \in D$,

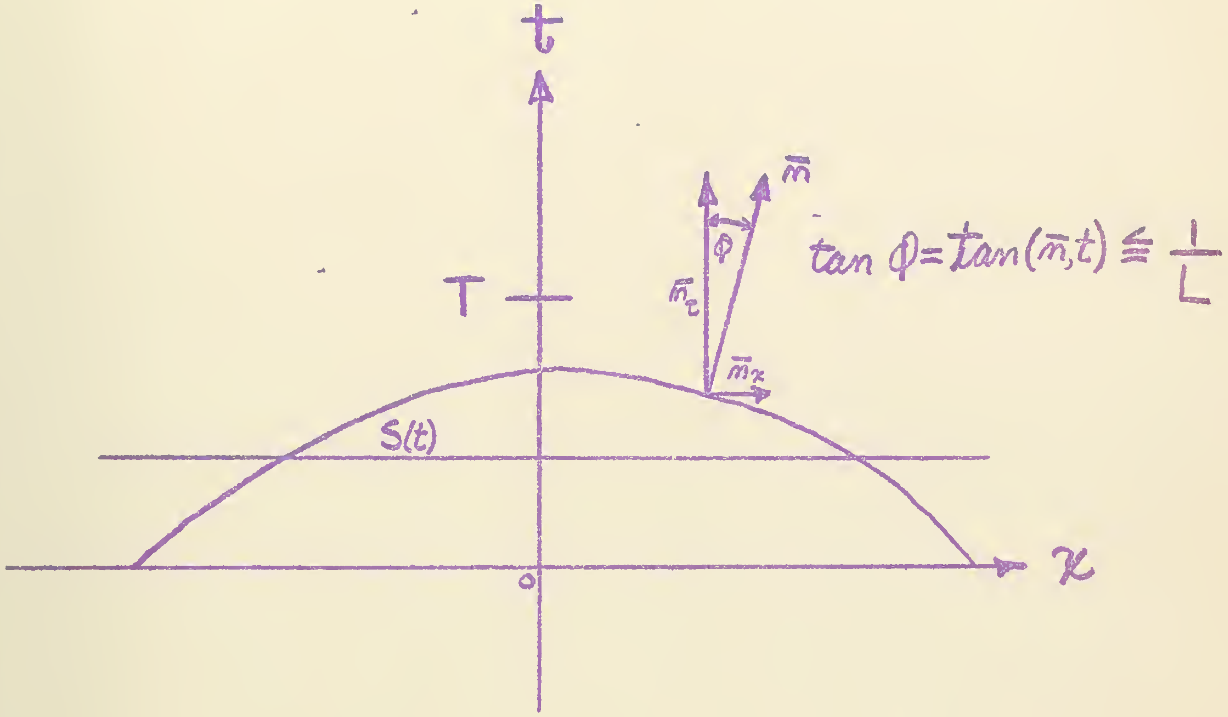

Figure #1. -- Sketch of Ω(T)

then $u(t,x) \equiv v(t,x)$ in the fundamental domain $\Omega(T)$.

Since u and v are both assumed to be solutions of (6), we may subtract one from the other to obtain

$$\frac{\partial(u-v)}{\partial t} = A(t,x,u)\frac{\partial u}{\partial x} + f(t,x,u) - A(t,x,v)\frac{\partial v}{\partial x} -$$

$$- f(t,x,v) + \left[A(t,x,u)\frac{\partial v}{\partial x} - A(t,x,u)\frac{\partial v}{\partial x}\right] \tag{10}$$

or

$$\frac{\partial(u-v)}{\partial t} = A(u)\frac{\partial(u-v)}{\partial x} + \left[A(u)-A(v)\right]\frac{\partial v}{\partial x} + f(u) - f(v)$$

where for brevity we write $A(t,x,u) = A(u)$, $A(t,x,v) = A(v)$, $f(t,x,u) = f(u)$, and $f(t,x,v) = f(v)$. Multiplying equation (10) from the right and from the left by the vector $(u-v)$ and adding the two results we obtain

$$\frac{\partial}{\partial t}|u-v|^2 = \left(A(u)\frac{\partial(u-v)}{\partial x}, u-v\right) + \left((u-v), A(u)\frac{\partial(u-v)}{\partial x}\right) + \tag{11}$$

$$+ 2\,\mathrm{re}\left(\left[A(u)-A(v)\right]\frac{\partial v}{\partial x}, u-v\right) + 2\,\mathrm{Re}\left(f(u)-f(v), u-v\right)$$

The last two terms on the right hand side of (11) are a result of the Hermitian type of symmetry assigned to the matrix A, and the definition of the scalar product employed.

If we now take the total derivative of the expression $\left(A(u)(u-v), u-v\right)$ we see that

$$\frac{d}{dx}\left\{\left(A(u)(u-v), u-v\right)\right\} = \left(\frac{dA(u)}{dx}(u-v), u-v\right) +$$

$$+ \left(A(u)\frac{\partial(u-v)}{\partial x}, u-v\right) + \tag{12}$$

$$+ \left(A(u)(u-v), \frac{(u-v)}{\partial x}\right)$$

But $\left(A(u)(u-v),\ \frac{\partial(u-v)}{\partial x}\right) =$

$$= \begin{pmatrix} a_{11}(u) & a_{21}(u) \\ a_{12}(u) & a_{22}(u) \end{pmatrix} \begin{pmatrix} u_1-v_1 \\ u_2-v_2 \end{pmatrix} \circ \begin{pmatrix} \overline{\dfrac{\partial(u_1-v_1)}{\partial x}} \\ \overline{\dfrac{\partial(u_2-v_2)}{\partial x}} \end{pmatrix}$$

$$= (u_1-v_1)\left[a_{11}(u)\ \overline{\frac{\partial(u_1-v_1)}{\partial x}} + a_{12}(u)\ \overline{\frac{\partial(u_2-v_2)}{\partial x}}\right] +$$

$$+ (u_2-v_2)\left[a_{21}(u)\ \overline{\frac{\partial(u_1-v_1)}{\partial x}} = a_{22}(u)\ \overline{\frac{\partial(u_2-v_2)}{\partial x}}\right]$$

while $\left((u-v),\ A(u)\ \frac{\partial(u-v)}{\partial x}\right) =$

$$= \left((u_1-v_1 \quad u_2-v_2)\right) \circ \overline{\begin{pmatrix} a_{11}(u) & a_{21}(u) \\ a_{12}(u) & a_{22}(u) \end{pmatrix}} \begin{pmatrix} \overline{\dfrac{\partial(u_1-v_1)}{\partial x}} \\ \overline{\dfrac{\partial(u_2-v_2)}{\partial x}} \end{pmatrix}$$

$$= (u_1-v_1)\left[a_{11}(u)\ \overline{\frac{\partial(u_1-v_1)}{\partial x}} + \overline{a_{21}(u)}\ \overline{\frac{\partial(u_2-v_2)}{\partial x}}\right] +$$

$$+ (u_2-v_2)\left[\overline{a_{12}(u)}\ \overline{\frac{\partial(u_1-v_1)}{\partial x}} + a_{22}(u)\ \overline{\frac{\partial(u_2-v_2)}{\partial x}}\right]$$

so that $\left(A(u)(u-v),\ \frac{\partial(u-v)}{\partial x}\right) = \left(u-v,\ A(u)\ \frac{\partial(u-v)}{\partial x}\right)$

and hence (12) may be rewritten

$$\left(A(u)\ \frac{\partial(u-v)}{\partial x},\ u-v\right) + \left(u-v,\ A(u)\ \frac{\partial(u-v)}{\partial x}\right) =$$

$$= \frac{d}{dx}\left(A(u)(u-v),\ u-v\right) - \left(\frac{dA(u)}{dx}(u-v),\ u-v\right) \tag{13}$$

From the condition of continuity imposed on the derivatives of a_{jk}, we know that the matrix $\frac{dA}{dx}$ is bounded in the fundamental domain so that we may assert that

$$\left| \left(\frac{dA(u)}{dx} (u-v), (u-v) \right) \right| \leqq M_1 \, |u-v|^2 \tag{14}$$

where M_1 is a positive constant.

Now by expanding the matrix product and noting that the absolute value of a sum (or difference) is less than or equal the sum of the absolute values of the individual terms, we see that

$$\begin{aligned}
&\left| \operatorname{Re} \left(\left(A(u) - A(v) \right) \frac{\partial v}{\partial x}, u-v \right) \right| \leqq \\
&\leqq \left| a_{11}(u) \frac{\partial v_1}{\partial x} + \operatorname{Re} a_{21}(u) \frac{\partial v_2}{\partial x} \right| \cdot |u_1 - v_1| + \\
&\quad + \left| a_{11}(v) \frac{\partial v_1}{\partial x} + \operatorname{Re} a_{21}(v) \frac{\partial v_2}{\partial x} \right| \cdot |u_1 - v_1| + \\
&\quad + \left| \operatorname{Re} a_{12}(u) \frac{\partial v_1}{\partial x} + a_{22}(u) \frac{\partial v_2}{\partial x} \right| \cdot |u_2 - v_2| + \\
&\quad + \left| \operatorname{Re} a_{12}(v) \frac{\partial v_1}{\partial x} + a_{22}(v) \frac{\partial v_2}{\partial x} \right| \cdot |u_2 - v_2| .
\end{aligned}$$

Again utilizing the boundedness of the elements a_{jk} and the derivatives involved, we see that

$$\begin{aligned}
&2 \left| \operatorname{Re} \left(\left(A(u) - A(v) \right) \frac{\partial v}{\partial x}, u-v \right) \right| \leqq \\
&B|u_1 - v_1| + C|u_2 - v_2| \leqq M_2 |u-v|^2
\end{aligned} \tag{15}$$

where B, C, and M_2 are positive constants.

From (3) we note that

$$2\left|\operatorname{Re}\left(f(u) - f(v),\ u-v\right)\right| \leqq 2|u-v|\ \Phi\left(|u-v|\right) =$$

$$= 2F\left(|u-v|\right)^2. \tag{16}$$

Then substituting (14), (15), and (16) into (11) we see that

$$\frac{\partial |u-v|^2}{\partial t} \leqq \frac{d}{dx}\left\{\left(A(u)(u-v),\ u-v\right)\right\} + M_3|u-v|^2 + 2F\left(|u-v|\right)^2 \tag{17}$$

where M_3 is a positive constant. Let us now integrate (17) over the fundamental domain $\Omega(t)$ for $0 < t < T$, by applying Green's formula. Taking this term by term, and observing that $u \equiv v$ for $t = 0$, we obtain the following:

$$\int_{\Omega} \frac{\partial |u-v|^2}{\partial t}\, dt\, dx = \int_{C} |u-v|^2\, \bar{n}_t\, dx =$$

$$= \int_{S(t)} |u-v|^2\, dx + \int_{E} |u-v|^2\, \bar{n}_t\, dE$$

where $\bar{n}_t = \cos(\bar{n}, t)$.

$$\int_{\Omega} \frac{d}{dx}\left\{\left(A(u)(u-v),\ u-v\right)\right\} dt\, dx =$$

$$= \int_{\Omega} \frac{d}{dx}\left\{H(u-v:1)\right\} dt\, dx = \int_{C} H(u-v:1)\bar{n}_x\, dt =$$

$$= \int_{S(t)} H(u-v:1)\bar{n}_x\, dt + \int_{E} H(u-v:1)\bar{n}_x\, dt = \int_{E} H(u-v:\alpha)dt.$$

where $\alpha = \bar{n}_x = \sin(\bar{n}, t) = \cos(\bar{n}, x)$.

$$\therefore \int_{S(t)} |u-v|^2\,dx + \int_E \Big[|u-v|^2 \cos(\bar{n},t) - \tag{18}$$

$$- H\Big(u-v:\ \cos(\bar{n},x)\Big)\Big]\,dE \leqq M_3 \int_0^t d\tau \int_{S(\tau)} |u-v|^2\,dx +$$

$$+ 2 \int_0^t d\tau \int_{S(\tau)} F(|u-v|^2)\,dx.$$

According to inequality (9) we may assert that $|u-v|^2 \cos(\bar{n},t) - H\big(u-v:\cos(\bar{n},x)\big) \geqq |u-v|^2 \big[\cos(\bar{n},t) - L \sin(\bar{n},t)\big] \geqq 0$, since $L \leqq \operatorname{ctn}(\bar{n},t) = \dfrac{\cos(\bar{n},t)}{\sin(\bar{n},t)}$.

Thus we have

$$\int_{S(t)} |u-v|^2\,dx \leqq M_3 \int_0^t d\tau \int_{S(\tau)} |u-v|^2\,dx + \tag{19}$$

$$+ 2 \int_0^t d\tau \int_{S(\tau)} F(|u-v|^2)\,dx.$$

Because $F(Z)$ is concave, we know that in an interval (a,b), where $Z_k \in (a,b)$ and $\sum l_k = 1$

$$\sum_{k=1}^{m} l_k F(Z_k) \leqq F\left(\sum_{k=1}^{m} l_k Z_k\right)$$

Multiplying and dividing by the length of the interval, say $b - a$, gives

$$\frac{1}{b-a} \sum_{k=1}^{m} l_k F(Z_k)(b-a) \leqq F\left(\frac{1}{b-a}\left\{\sum_{k=1}^{m} l_k Z_k (b-a)\right\}\right)$$

Then if we let m increase indefinitely, we see that $l_k(b-a)$ becomes a k-th subinterval of $(b-a)$ and each side is a Riemann sum, so that

$$\frac{1}{b-a} \int F(Z)\,dx \leqq F\left(\frac{1}{b-a} \int Z\,dx\right). \tag{20}$$

Applying (20) to equation (19) we see that

$$\int_{S(t)} |u-v|^2\, dx \leqq M_3 \int_0^t d\tau \int_{S(\tau)} |u-v|^2\, dx +$$

$$+ 2 \int_0^t \sigma(\tau) F\left(\frac{1}{\sigma(\tau)} \int_{S(\tau)} |u-v|^2\, dx\right) d\tau \tag{21}$$

where $\sigma(\tau)$ is the length or measure of the line segment $S(\tau)$.

Without loss of generality we may assume that $\sigma(\tau) \leqq 1$. Again taking into account the fact that $F(Z)$ is concave, we obtain that

$$\sigma(\tau)F(x_1) + \left(1-\sigma(\tau)\right) F(x_2) \leqq F\left(x_1\sigma(\tau)\right) + x_2\left(1-\sigma(\tau)\right) .$$

If now we let $x_2 = 0$, and observe that $F(0) = 0$ we note that

$$\sigma(\tau)F(Z) \leqq F\left(\sigma(\tau)Z.\right) \tag{22}$$

If now we let $Z(t) = \int_{S(t)} |u-v|^2\, dx$ and apply (22) to equation (21) we find

$$Z(t) \leqq M_3 \int_0^t Z(\tau)d\tau + 2 \int_0^t F\left(Z(\tau)\right) d\tau \tag{23}$$

and the proposition will be proved if we show $Z(t) \equiv 0$ for $0 \leqq t \leqq T$. Let us suppose the contrary. We know that $Z(0) = 0$. Therefore there must exist two numbers $\alpha \geqq 0$ and $\beta \geqq 0$, such that

(a) $Z(t) = 0$ when $0 \leqq t \leqq \alpha$.

(b) $0 < Z(t) < 1$ when $\alpha < t < \beta$.

Then for $\alpha < \tau < \beta$, we see from equation (22) that

$F(1)Z(\tau) \leqq F\left(Z(\tau)\right)$, and hence

$$Z(t) \leqq \frac{M_3}{F(1)} \int_0^t F\big(Z(\tau)\big)\, d\tau +$$

$$+ 2\int_0^t F\big(Z(\tau)\big)\, d\tau = M_4 \int_0^t F\big(Z(\tau)\big)\, d\tau ,$$

or $$Z(t) \leqq M_4 \int_\alpha^t F\big(Z(\tau)\big)\, d\tau ,\ \alpha < t < \beta , \tag{24}$$

where since F is an increasing function, $F(1) > 0$.

Now put $\xi(t) = M_4 \int_\alpha^t F\big(Z(\tau)\big)\, d\tau$. Then, since $F(Z)$ is an increasing function we have:

$$\frac{d\,\xi(t)}{dt} = M_4\, F\big(Z(t)\big) \leqq M_4\, F\big(\xi(t)\big) . \tag{25}$$

Evidently $$\int_0^{\xi(\beta)} \frac{d\xi}{F(\xi)} \leqq M_4(\beta - \alpha) \tag{26}$$

which contradicts the divergence of the integral

$$\int_0^{\delta} \frac{d\xi}{F(\xi)} ,\ 0 < \delta . \quad \text{QED.}$$

Thus the uniqueness has been demonstrated.

To demonstrate the practical applicability of these uniqueness conditions let us consider briefly Maxwell's wave equations. Of course, solutions to Maxwell's equations have been attained in other ways, (c. f. Courant and Hilbert [6]), but this will serve to illustrate our principles. Maxwell's wave equations are:

$$(1) \qquad \nabla \times \vec{H} = \frac{1}{C}\frac{\partial \vec{E}}{\partial t}$$

$$(2) \qquad \nabla \times \vec{E} = -\frac{1}{C}\frac{\partial \vec{H}}{\partial t}$$

$$(3) \qquad \nabla \cdot \vec{H} = \nabla \cdot \vec{E} = 0$$

where $\vec{E} = \vec{E}(x,y,Z,t)$ is the electric field strength vector, $\vec{H} = \vec{H}(x,y,Z,t)$ is the magnetic field strength vector and C = constant velocity of light in vacuum = 3×10^{10} cm/sec.

Obviously for our purposes we need consider only system (1). In component form we have

$$\frac{\partial E_1}{\partial t} = C\frac{\partial H_3}{\partial x_2} - C\frac{\partial H_2}{\partial x_3}$$

$$\frac{\partial E_2}{\partial t} = C\frac{\partial H_1}{\partial x_3} - C\frac{\partial H_3}{\partial x_1}$$

$$\frac{\partial E_3}{\partial t} = C\frac{\partial H_2}{\partial x_1} - C\frac{\partial H_1}{\partial x_2} .$$

Denoting E_1, E_2, and E_3 by u_1, u_2, and u_3 respectively and H_1, H_2, H_3 by u_4, u_5, u_6 respectively, Maxwell's equations take the form of system (1). We see that our coefficient matrices will then be as follows:

$$A^{(1)} = (a_{jk}^{(1)}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -C \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^{(2)} = (a_{jk}^{(2)}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C & 0 & 0 \end{pmatrix}$$

$$A^{(3)} = (a_{jk}^{(3)}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -C & 0 \\ 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and in matrix form,

$$\left(\frac{\partial u}{\partial t}\right) = \left(\sum_{i=1}^{n} A^{i} \frac{\partial u}{\partial x_i}\right) \tag{27}$$

If now we multiply both sides by $\sqrt{-1}$ I our matrix A takes on the desired Hermitian type of symmetry and we are ready to test our conditions.

1. The system does have the required symmetry.

2. The functions $a_{jk}^{(i)}$, being constants, are continuous and do have continuous derivatives in any prescribed domain, with respect to all variables.

3. Since $f_k \equiv 0$, the inequality

$$|f(t,x,u) - f(t,x,v)| \leqq \Phi(|u-v|)$$

is satisfied for $\Phi(|u-v|) = |u-v|$, say.

4. Then $F(Z) = \sqrt{Z}\,\Phi(\sqrt{Z}) = Z$ is a positive increasing concave function such that

$$\int_0^{\delta} \frac{dZ}{F(Z)} = \int_0^{\delta} \frac{dZ}{Z} \to \infty \quad (0 < \delta).$$

Although in the case of Maxwell's equations as stated, the vector-function f is identically zero, we note that the uniqueness of the solution is not destroyed even though a perturbation effect is present, so long as the perturbing function satisfies the inequality conditions of 3 and 4.

We will conclude with a statement of the conditions required for uniqueness of the solution of a symmetrical system of second order, which in matrix notation has the form

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} +$$

$$+ f\left(t,x,u,\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \cdots, \frac{\partial u}{\partial x_n}\right). \qquad (28)$$

The proof of this uniqueness can be found in reference work [2], and is very similar to the proof for the first order system.

The symmetry property appears in this statement, that the matrix

$$A_{ij} = A_{ij}(t,x,u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n})$$

is Hermitian, and $A_{ij} = A_{ji}$. It is further assumed that for any n-dimensional vector ξ,

$$\mu^2 \sum_{i=1}^{n} |\xi_i|^2 \leqq \sum_{i,j=1}^{n} (A_{ij}\xi_i \xi_j) \qquad (29)$$

where $\mu \neq 0$ is a real number. From (29) follows the fact that for any real α_i, the characteristic equation

$$\left| \sum_{i,j=1}^{n} A_{ij}\alpha_i \alpha_j - \lambda I \right| = 0$$

has only real roots, and hence system (28) is hyperbolic.

Let the following conditions be satisfied:

1. The matrix A is continuous and is continuously differentiable with respect to the variables

$$x_1, \cdots, x_n, u_1, \cdots, u_N, \frac{\partial u_1}{\partial t}, \cdots, \frac{\partial u_N}{\partial t}, \frac{\partial u_1}{\partial x_1}, \cdots, \frac{\partial u_N}{\partial x_n}$$

2. The vector function f satisfies the inequality

$$\left| f(t,x,u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}) - \right.$$

$$\left. - f(t,x,v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x_1}, \cdots, \frac{v}{x_n}) \right| \leqq$$

$$\leqq \Phi \left\{ \left(|u-v|^2 + \sum_{k=0}^{n} \left| \frac{\partial u}{\partial x_k} - \frac{\partial v}{\partial x_k} \right|^2 \right)^{1/2} \right\}$$

where $\frac{\partial u}{\partial x_0} \equiv \frac{\partial u}{\partial t}$.

3. $F(Z) = \sqrt{Z}\ \Phi\ (\sqrt{Z})$ is a positive, non-decreasing, and concave function such that

$$\int_0^{\delta} \frac{dZ}{F(Z)} \text{ diverges } \quad (0 < \delta).$$

Then if u and v are two twice-continuously differentiable solutions of system (28) which satisfy the initial conditions

$$u\Big|_{t=0} = v\Big|_{t=0}$$

$$\frac{\partial u}{\partial t}\Big|_{t=0} = \frac{\partial v}{\partial t}\Big|_{t=0}$$

these solutions must coincide in some fundamental domain of system (28). As a final note, this uniqueness theorm is valid in particular, for a single quasi-linear hyperbolic equation of the second order. Thus the work of S. L. Soboleff [7] in regard to obtaining a priori values for single hyperbolic equations of second order can perhaps be extended analogously to such symmetrical systems.

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